RMIT UNIVERSITY

LT1.2: LAPLACE TRANSFORMS

SOLVING DIFFERENTIAL EQUATIONS

Example

Given the following first order differential equation, $\frac{dy}{dt} + y = 3e^{2t}$, where $y(0) = 4$.

Find $y(t)$ *using Laplace Transforms.*

Sol$^{n:}$

To begin solving the differential equation we would start by taking the Laplace transform of both sides of the equation.

$$L\left[\frac{dy}{dt} + y\right] = L\left[3e^{2t}\right]$$

Taking the Laplace Transform of both sides of the equation.

$$L\left[\frac{dy}{dt}\right] + L[y] = 3L\left[e^{2t}\right]$$

Separating terms.

$$sY - y(0) + Y = 3 \times \frac{1}{s-2}$$

Transforms as derived from tables.

$$sY - 4 + Y = \frac{3}{s-2}$$

Substituting for $y(0) = 4$

$$Y(s+1) = \frac{3}{s-2} + 4$$

Taking Y as a common factor.

$$Y = \frac{3}{(s-2)(s+1)} + \frac{4}{(s+1)}$$

$$Y = \frac{3}{(s+1)(s-2)} + \frac{4(s-2)}{(s+1)(s-2)}$$

$$Y = \frac{4s-5}{(s-2)(s+1)}$$

Making Y the subject.

Use partial fractions to expand $\frac{4s-5}{(s-2)(s+1)}$

$$\therefore \qquad \frac{4s-5}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1}$$

$$4s - 5 = A(s+1) + B(s-2)$$

By selecting appropriate values of *s*, we can solve for *A* & *B*.

Letting *s* = -1, and substituting into the above equation gives

$$4(-1)-5 = A(-1+1)+B(-1-2)$$
$$-4-5 = A(0)+B(-3)$$
$$-9 = -3B$$
$$B = \frac{-9}{-3} = 3$$

Now let *s* = 2, and substitute into the same equation

$$4(2)-5 = A(2+1)+B(2-2)$$
$$8-5 = A(3)+B(0)$$
$$3 = 3A$$
$$A = \frac{3}{3} = 1$$

So

$$\frac{4s-5}{(s-2)(s+1)} = \frac{1}{s-2}+\frac{3}{s+1}$$

Therefore

$$Y = \frac{1}{s-2}+\frac{3}{s+1}$$

To obtain a solution $y(t)$ to the differential equation from $Y(s)$ we need to find the inverse Laplace transform of Y.

$$\therefore \quad L^{-1}[Y] = L^{-1}\left[\frac{1}{s-2}+\frac{3}{s+1}\right]$$

$$\therefore \quad y(t) = L^{-1}\left[\frac{1}{s-2}\right]+3L^{-1}\left[\frac{1}{s+1}\right]$$

$$\therefore \quad y(t) = e^{2t}+3e^{-t}$$

Inverse transforms obtained from tables.

Example

Given the following first order differential equation, $y'' + y' = 5\cos 2t$; $y(0) = 0$; $y'(0) = 0$

Find $y(t)$ *using Laplace Transforms.*

Sol$^{\underline{n}}$:

$$L[y''] + L[y'] = L[5\cos 2t]$$

$$s^2 Y - sy(0) - y'(0) + sY - y(0) = \frac{5s}{s^2 + 2^2}$$

$$Y(s^2 + s) = \frac{5s}{s^2 + 4}$$

$$Y = \frac{5s}{(s^2 + s)(s^2 + 4)}$$

$$Y = \frac{5s}{(s)(s+1)(s^2+4)} = \frac{5}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs + C}{s^2 + 4}$$

$\therefore \quad 5 = A(s^2 + 4) + (Bs + C)(s + 1)$

An alternative method for solving the unknowns *A, B, & C* in the above equation is called "*equating coefficients of powers of s":*

	LHS	=	RHS	
s^0 :	5	=	$4A + C$	eq$^{\underline{n}}$ 1.
s^1 :	0	=	$B + C$	eq$^{\underline{n}}$ 2.
s^2 :	0	=	$A + B$	eq$^{\underline{n}}$ 3.

From eq$^{\underline{n}}$ 3 $\quad A = -B$

From eq$^{\underline{n}}$ 2 $\quad C = -B$

Substitute in eq$^{\underline{n}}$ 1 $\quad 5 = 4(-B) + (-B) = -5B$

$\therefore \quad A = 1, \quad B = -1, \quad C = 1$

$$\therefore \quad Y = \frac{1}{s+1} + \frac{-s+1}{s^2+4} = \frac{1}{s+1} - \frac{s}{s^2+4} + \frac{1}{s^2+4}$$

$$\therefore \quad y(t) = L^{-1}[Y] = L^{-1}\left[\frac{1}{s+1}\right] - L^{-1}\left[\frac{s}{s^2+4}\right] + L^{-1}\left[\frac{1}{s^2+4}\right]$$

From tables:

$$y(t) = e^{-t} - \cos 2t + \sin 2t$$